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DETERMINATION OF NON-STANDARD INPUT SIGNAL MAXIMIZING THE ABSOLUTE ERROR

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Abstract

The paper presents a method and algorithm for determining input signals which maximize the absolute value of error. Being maximum, the values of these errors are valid for any dynamic signal which might occur at the input of a real system. In this way all the possible signals are taken into consideration at the same time. It should be stressed that these can be non-determined signals whose form cannot be predicted a priori. Two types of signals are taken into consideration: signals with a magnitude constraint and signals constrained in magnitude and rate of change. Solutions derived in the paper enable calculation of the absolute value of error by means of analytical formulae which give precise results and can be realised in a very short time.

Keywords: maximum dynamic error, dynamic measurement, signal constraints.

1. Introduction

In measurement of dynamic signals the absolute value of error is of essential importance, especially for tracking systems and systems intended for measurement of signal shape. Such systems are commonly applied in many different branches, *e.g.* in electrical metrology (transducers, filters, strain gauge amplifiers, measuring microphones *etc.*), in geodesy (accelerometers, vibrometers *etc.*), in medicine (electroencephalographs, cardiographs *etc.*), in meteorology (autocomparators, autobridges *etc.*). For systems of such type input signals are unknown and can not be determined in advance. Also the rated operation conditions of these systems are very difficult to define as they usually work in a dynamic mode, far from a steady state. For that reason there is no point in determining the errors by means of typical standard signals as the results received depend essentially on the input signals for which they are computed. Moreover, in practice real systems are not excited by standard signals, but usually by unknown dynamic signals which are decidedly different from the standard ones. It should be noted, however, that the solution of a problem posed in a way which could make the error values independent of the input signal form is possible for maximum errors. But the procedure of determination of maximum errors requires special input signals to be used, which warrant that the error values determined with them will always be higher or at least equal to the value generated by any other signal. Below we will determine shapes of such signals constrained in magnitude, and simultaneously in magnitude and rate of change.

2. General assumption

Let the mathematical model of a calibrated system be given by a state equation

$$\dot{x}_m(t) = \mathbf{A}_m x(t) + \mathbf{B}_m u(t) \qquad x_m(0) = 0$$
$$y_m(t) = \mathbf{C}_m^T x(t) \tag{1}$$

and the system constituting its standard be given by a similar equation

$$\dot{x}_r(t) = \mathbf{A}_r x(t) + \mathbf{B}_r u(t) \qquad x_r(0) = 0$$
$$y_r(t) = \mathbf{C}_r^T x(t). \tag{2}$$

Let us introduce a new state equation

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}^T x(t) \tag{3}$$

in which

$$x(t) = \begin{bmatrix} x_r(t) \\ x_m(t) \end{bmatrix} \mathbf{A} = \begin{bmatrix} A_r & 0 \\ 0 & A_m \end{bmatrix} \mathbf{B} = \begin{bmatrix} B_r \\ B_m \end{bmatrix} \mathbf{C} = \begin{bmatrix} C_r \\ -C_m \end{bmatrix}, \tag{4}$$

where in (1)-(4) $u(t)$ and $y(t)$ are input and output respectively, $x(t)$ is state vector, **A**, **B**, **C** are real matrices of corresponding dimensions.

3. Shape of signals with one constraint

Below, for the output $y(t)$ Eq. (3) presenting the error between the systems (1) and (2) we shall determine the input signal $u(t) = u_0(t)$ constrained in magnitude

$$|u(t)| \le a \tag{5}$$

which maximizes its value over the interval $[0, T]$. For this purpose let us write the error in the form of a convolution integral

$$\underset{t\in[0,T]}{y(t)} = \int_0^t k(t-\tau)u(\tau)d\tau \, , \tag{5}$$

where

$$k(t) = \mathbf{C}^T e^{\mathbf{A}t}\mathbf{B} \, . \tag{7}$$

From (6) it directly results that the maximum value of $|y(t)|$ occurs for $t = T$

$$\max_{t\in[0,T]} |y(t)| = y(T) \tag{8}$$

if

$$u_0(\tau) = a \cdot sign[k(T-\tau)], \tag{9}$$

where a is magnitude, and

$$\max_{t\in[0,T]} |u_0(\tau)| = a \, . \tag{10}$$

Changing τ for t we can write

$$|y(T)| = \int_0^T k(T-t)u(t)\,dt \,. \tag{11}$$

And $u_0(t)$ maximizing (11) has now the form

$$u_0(t) = a \cdot sign[k(T-t)] \tag{12}$$

which, for t_n switchings in $t_1, t_2, ..., t_n$ and the assumption that the first switching occurs from $+a$ to $-a$, can be determined by means of the following relation

$$u_0(t_1, t_2, ..., t_n) = a \cdot \sum_{i=0}^{n} (-1)^i \cdot \int_{t_i}^{t_{i+1}} k(T-t)dt \,, \tag{13}$$

where $t_0 = 0,\ t_{i+1} = T \quad for \quad i = n, \quad n$ - number of switchings.
Substitution of (12) into (11) gives finally

$$|y(T)| = a \cdot \int_0^T |k(T-t)|\,dt = a \cdot \int_0^T |k(t)|\,dt \tag{14}$$

which is not difficult to compute.

4. Shape of signals with two constraints

Let us suppose now that on signals $u(t)$ two simultaneous constraints with respect to their magnitude (5) and rate of change ϑ (15) are imposed

$$\max_{t\in[0,T]} |\dot{u}(t)| \le \vartheta \,. \tag{15}$$

Let us present $u(t)$ by means of the integral

$$u(t) = \int_0^t \varphi(\tau)d\tau \,, \tag{16}$$

then the error can be written in the following form

$$\underset{t\in[0,T]}{y(T)} = \int_0^T k(T-t) \int_0^t \varphi(\tau)d\tau\, dt \tag{17}$$

and constraints (5) and (15) referring to $u(t)$ for function $\varphi(\tau)$ are now as follows

$$\max_{t\in[0,T]} \left| \int_0^t \varphi(\tau)d\tau \right| = \max_{t\in[0,T]} |u(t)| \le a \,, \tag{18}$$

and

$$\max_{t\in[0,T]} |\varphi(t)| = \max_{t\in[0,T]} |\dot{u}(t)| \le \vartheta \,. \tag{19}$$

Changing the integration order in (17), we have

$$\underset{t\in[0,T]}{y(T)} = \int_0^T \varphi(\tau)\int_\tau^T k(T-t)dt\,d\tau \tag{20}$$

and after replacing τ for t, we get finally

$$\underset{t\in[0,T]}{y(T)} = \int_0^T \varphi(t)\int_t^T k(T-\tau)d\tau\,dt\,. \tag{21}$$

From (21), it is evident that $\varphi(t)$, which maximizes $y(T)$, has the maximum magnitude $\varphi(t) = \pm\vartheta$ by virtue of the formula (19) if

$$\varphi(t) = sign\int_t^T k(T-\tau)d\tau \tag{22}$$

and $\varphi(t) = 0$, in such subintervals, for which the resulting form (22) between the switching moments is

$$\left|\int_0^t \varphi(\tau)d\tau\right| > a\ \ . \tag{23}$$

Using equations (15) - (23) we can determine signal $u(t) = u_0(t)$ in the following cases:

First case

If $|\int_0^t f_0(\tau)d\tau \neq a|$ for δ varying in the intervals $[0,+\vartheta]$ and $[0,-\vartheta]$, (Fig. 1 and Fig. 2), where $f_0(t) = \pm\vartheta$ for $\int_t^T k(T-\tau)d\tau > +\delta$ and $\int_t^T k(T-\tau)d\tau < -\delta$ respectively, then the signal $u_0(t)$ is determined in three following steps, according to Eqs. (4.10)-(4.16).

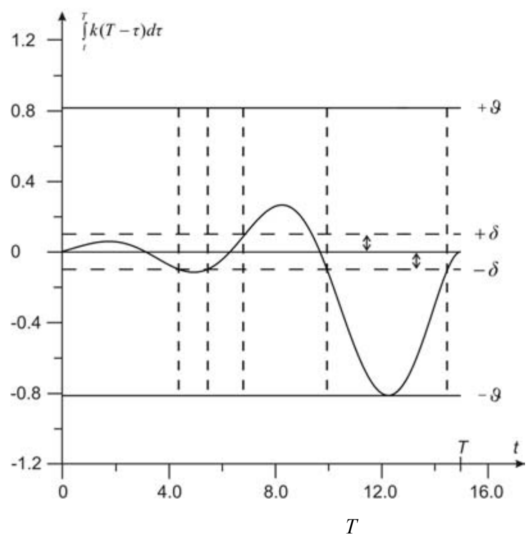

Fig. 1. Exemplary function $\int_t^T k(T-\tau)d\tau$.

During the first step, the 'bang-bang' functions $f_1(t)$ of the magnitude $\pm\vartheta$ are determined with switching moments resulting from (24) - Fig. 3.

$$f_1(t) = +\vartheta \quad if \quad \varphi(t) > 0$$
$$f_1(t) = -\vartheta \quad if \quad \varphi(t) < 0 \tag{24}$$

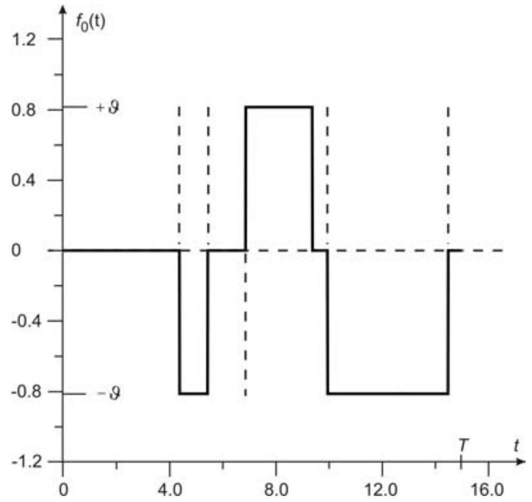

Fig. 2. Constraints resulting from $\int_t^T k(T-\tau)d\tau > +\delta$ and $\int_t^T k(T-\tau)d\tau < -\delta$.

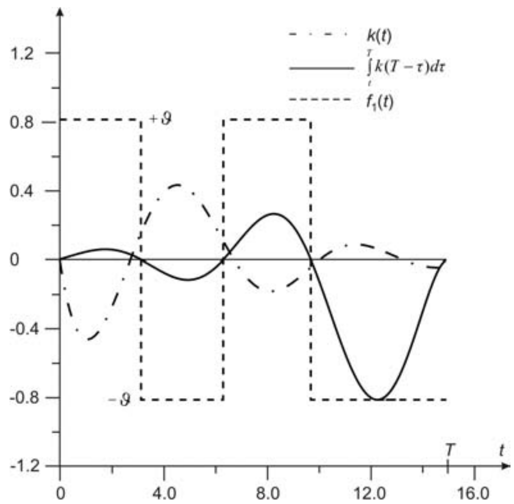

Fig. 3. Exemplary functions $k(t)$, $\int_t^T k(T-\tau)d\tau$ and $f_1(t)$.

In the second step, we obtain the function $f_2(t)$ by integrating $f_1(t)$ - Fig. 4.
Function $f_2(t)$ at particular switching intervals $t_1, t_2, \ldots, t_n$ of $f_1(t)$ is given by the following relations for $t \le t_1$, $n = 1$

$$f_2(t) = \vartheta \cdot t \ , \tag{25}$$

for $t_1 < t \le t_2$, $n = 2$

$$f_2(t) = \vartheta \cdot t_1 - \vartheta \cdot (t - t_1) \;, \tag{26}$$

for $t_i < t \le t_{i+1}$, $i = 2,3,...,n$, $t_{n+1} = T$, $n-$ number of switchings

$$f_2(t) = \vartheta \cdot t_1 + \vartheta \cdot \left[\sum_{j=2}^{i} (-1)^{j-1} \cdot (t_j - t_{j-1}) \right] + (-1)^i \cdot \vartheta \cdot (t - t_i) \,. \tag{27}$$

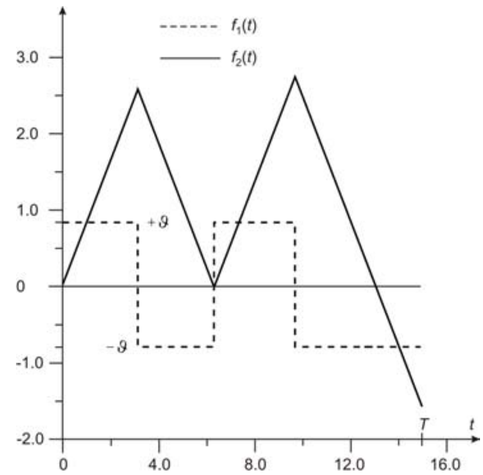

Fig. 4. Functions $f_1(t)$ and $f_2(t) = \int_0^t f_1(\tau) d\tau$.

In the last step, we determine the function $f_3(t)$ on the basis of $f_2(t)$. Relations are as follows

$$\begin{aligned} f_3(t) &= \pm\vartheta \;\; if \;\; \left| f_2(t) \right| \le a \\ f_3(t) &= 0 \quad if \;\; \left| f_2(t) \right| > a. \end{aligned} \tag{28}$$

Finally, we obtain the signal $u(t) = u_0(t)$ through integration of $f_3(t)$ and this is the aim. The operation is shown in Fig. 5.

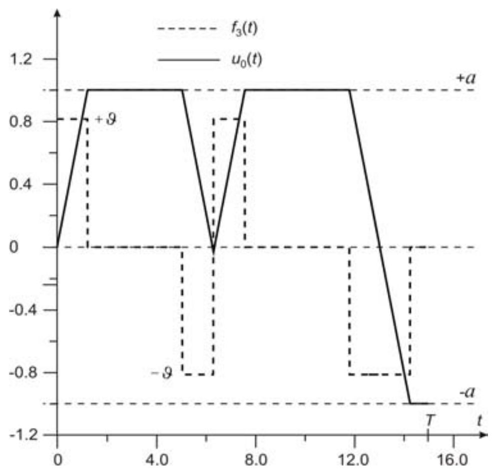

Fig. 5. Function $f_3(t)$ and signal $u_0(t) = \int_0^t f_3(\tau) d\tau$.

During the intervals in which $f_3(t) = \pm\vartheta$, the signal waveform is triangular, with the slope of $\pm\vartheta$. In the intervals when $f_3(t) = 0$, the signal is a constant of the magnitude $\pm\ a$.

For n switching moments of $f_3(t)$ the value of error is described by the following equations:
for $n = 1$

$$y(T) = \frac{h_1}{t_1}\int_0^{t_1} k(T-\tau)\tau\, d\tau \ + \ \frac{h_T - h_1}{T - t_1}\int_{t_1}^{T} k(T-\tau)\,(\tau - t_1)\, d\tau \ + h_1\int_{t_1}^{T} k(T-\tau)\, d\tau \ , \tag{29}$$

for $n \geq 2$

$$\begin{aligned} y(T) = \frac{h_1}{t_1}\int_0^{t_1} k(T-\tau)\tau\, d\tau + \sum_{i=2}^{n}\Big[\frac{h_i - h_{i-1}}{t_i - t_{i-1}}\int_{t_{j-1}}^{t_j} k(T-\tau)(\tau - t_{i-1})\, d\tau \\ + h_{i-1}\int_{t_{j-1}}^{t_j} k(T-\tau)\, d\tau\Big] \ + \ \frac{h_T - h_n}{T - t_n}\int_{t_n}^{T} k(T-\tau)\,(\tau - t_n)\, d\tau \ + h_n\int_{t_n}^{T} k(T-\tau)\, d\tau, \end{aligned} \tag{30}$$

where $h_i = u_0(t_i),\ h_T = u_0(T)$.
Fig. 6 presents the signal $u_0(t)$ and the error $y(t)$ corresponding to it.

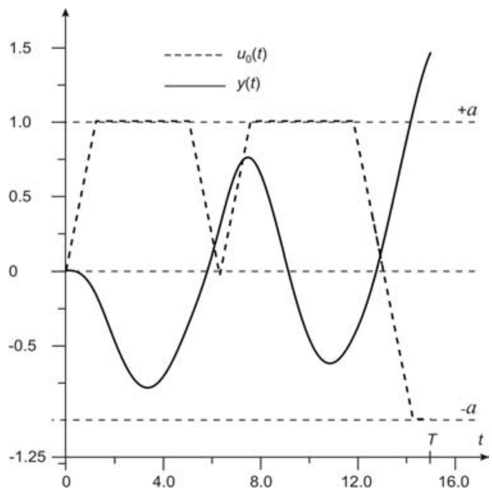

Fig. 6. Signal $u_0(t)$ and error $y(t)$.

Second case

If $\vartheta \cdot T \leq a$**,** signal $u_0(t)$ is given directly by

$$u_0(t) = \vartheta \cdot \int_0^t sign[k(T-\tau)]d\tau \tag{31}$$

and the error equals

$$y(T) = \int_0^T k(t-\tau)u_0(\tau)d\tau\ . \tag{32}$$

Fig. 7 presents the signal $u_0(t)$ and error $y(t)$ corresponding to it.

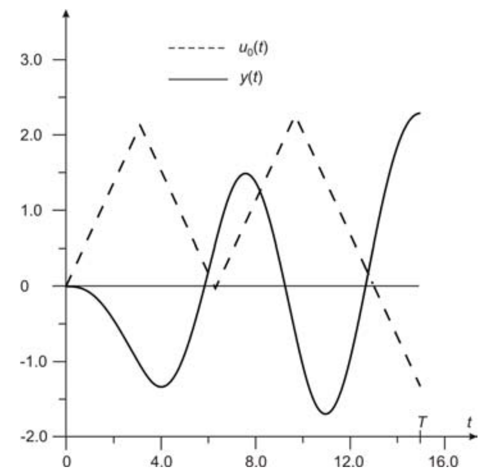

Fig. 7. Signal $u_0(t)$ and error $y(t)$.

Third case

If $\vartheta \cdot T > a$ then the signal $u_0(t)$ is determined indirectly by means of the functions $f_4(t)$-$f_6(t)$

$$f_4(t) = \frac{k(t)\cdot\vartheta}{2a}, \tag{33}$$

$$f_5(t) = a \cdot sign[f_4(T-t)], \tag{34}$$

$$\begin{aligned} f_6(t) &= f_5(t) \quad if \quad t < \frac{2\cdot a}{\vartheta} \\ f_6(t) &= f_5(t) - f_5\left(t - \frac{2\cdot a}{\vartheta}\right) \quad if \quad \frac{2\cdot a}{\vartheta} \le t < T. \end{aligned} \tag{35}$$

The functions $f_4(t)$ and $f_5(t)$ are shown in Fig. 8, while $f_6(t)$ and the signal $u_0(t) = \int_0^t f_6(\tau)\,d\tau$ in Fig. 9.

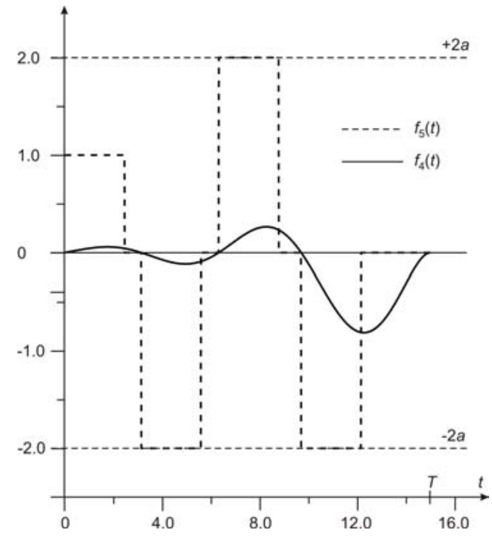

Fig. 8. Functions $f_4(t)$ and $f_5(t)$.

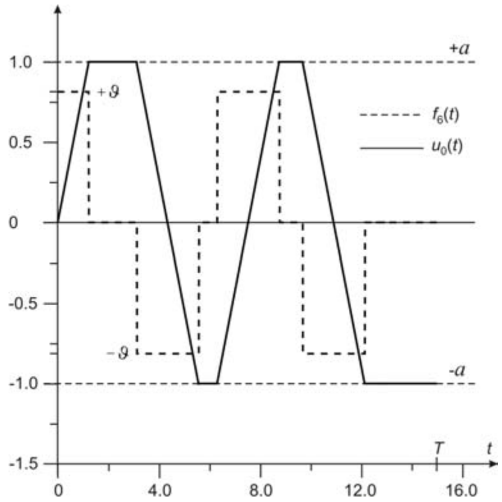

Fig. 9. Function $f_6(t)$ and signal $u_0(t)$.

Fig. 10 shows the signal $u_0(t)$ and the error $y(t)$ corresponding to it.

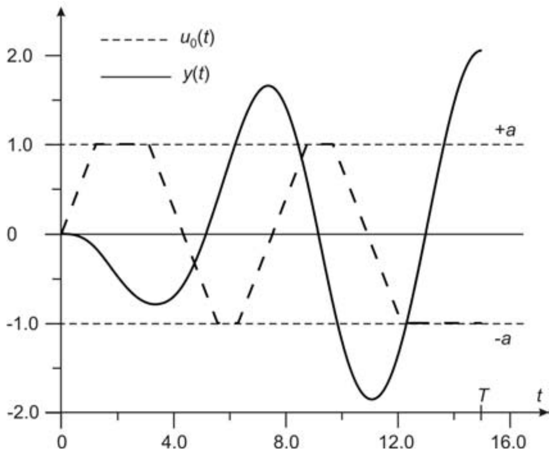

Fig. 10. Signal $u_0(t)$ and error $y(t)$.

5. Conclusion

The paper presents a method for determining signals which maximize the absolute value of error. The error determined by means of such signals corresponds to the errors of maximum value describing the classes of accuracy in the case of instruments intended for static measurement. Therefore the solutions presented in the paper can find practical application in calibration of different measuring systems intended for measurement of dynamic signals. Particularly these signals can be non-determined whose form we are not able to predict in advance. The value of the error calculated by means of formulae derived in this paper is precise, and can be received in a very short calculation time. It is due to the limit of the possible solutions to signals of rectangular, triangular or trapezoid shapes.

References

[1] B. Bartosinski, W. Toczek: “Some methods of diagnosis of analog circuits using mixed signal test bus IEEE 1149.4”. *Metrol Meas Syst*, vol. X, no. 2, 2003, pp. 157-172.

[2] I. Kricevskij: “O nakoplenii vozmuscenij v linejnych sistemach”. *Teoria funkcji, funkcjonalnyj analiz i ich prilozenija*”. vol. 3, 1967.

[3] I. Kricevskij, G.M. Ulanov: “Ob efektivnom obobscenii teorii nakoplenia otklonenij”. *Prikladnaja matematika i mechnaika*, vol. 37, 1973.

[4] E. Layer: “Theoretical Principles for Establishing a Hierarchy of Dynamic Accuracy with the Integral-Square-Error as an Example”. *IEEE Trans. Inst. Meas.,* vol. 46, no. 5, 1997, pp. 1178-1182.

[5] E. Layer: “Mapping Error of Linear Dynamic Systems Caused by Reduced-Order Model”. *IEEE Trans. Inst. Meas.,* vol. 50, 2001, pp.792-800.

[6] E. Layer: „Przestrzeń rozwiązań sygnałów maksymalizujących kwadratowy wskaźnik jakości”. *Mat. Konf. Modelowanie i symulacja systemów pomiarowych.* AGH, Krynica, 2001, pp.25-28. (in Polish)

[7] E. Layer: “Shapes of Input Signals for Calibration of Measuring Systems Intended for the Measurement of Dynamic Signals”. *Proc. IMEKO TC7 Symp*., Cracow, 2002, pp. 146-149.

[8] E. Layer: *Modelling of Simplified Dynamical Systems*. Springer-Verlag, Berlin, Heidelberg New York, 2002.

[9] E. Layer: “Non-standard input signals for the calibration and optimization of the measuring systems”. *Measurement,* vol.34, issue 2, 2003, pp.179-186.

[10] M. Ljubojevicz**: “**Suboptimal input signal for linear system identification”. *Int. J. Control 17*, 1973, pp. 659-669.

[11] N.K. Rutland: “The Principle of Matching: Practical Conditions for Systems with Inputs Restricted in Magnitude and Rate of Change”. *IEEE Trans. Autom. Control,* vol. 39, 1994, pp. 550-553.

[12] K. Tomczyk: “Optymalizacja parametrow matematycznych modeli wzorców ze względu na transformacje nieznieksztalcająca”. *Mat. V Sympozjum. nt. Pomiarow Dynamicznych, Szczyrk, Prace Komisji Metrologii PAN,* Katowice, 2005, pp. 119-128. (in Polish)

[13] K. Tomczyk: “Application of genetic algorithm to measurement system calibration intended for dynamic measurement”. *Metrol Meast Syst,* vol. XIII, no. 1, 2006, pp. 193-203.

[14] K. Tomczyk: “Computer aided system for determining maximum dynamic errors of chosen measuring instruments”. *PhD thesis*, Cracow University of Technology, Cracow, 2006.

[15] V. Zakian: “Critical Systems and Tolerable Inputs”. *Int. J. Control,* vol.49, 1989, pp. 1285-1289.

[16] V. Zakian: “Perspectives of the Principle of Matching and the Method of Inequalities”. *Int. J. Control,* vol. 65, 1996, pp.147-175.